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Making electromagnetic wavelets: II. Spheroidal shell antennas

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Abstract

In the companion paper, a charge–current distribution was obtained for radiating electromagnetic wavelets. However, it cannot be realized because of two drawbacks: (a) it requires the existence of magnetic charges, and (b) the charge–current distribution, which is concentrated on a spheroidal surface S_α , is too singular for practical implementation. Both of these difficulties are resolved here. The first is resolved by using Hertz vectors to generate a charge–current distribution on S_α due solely to bound electric charges. The second is resolved by replacing S_α with a spheroidal shell of finite thickness. This generalizes the usual boundary conditions on an interface between electromagnetic media by allowing the transition to be gradual.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

This paper is aimed at physicists and mathematical physicists interested in classical electrodynamics and optics. In the companion paper [K4], a coherent charge–current distribution for radiating electromagnetic wavelets was constructed on an oblate spheroidal surface S_α . Its main drawback was the necessity of including magnetic along with electric charges, making the sources impossible to realize. Here we show how this difficulty can be overcome by using Hertz vectors [BW99, J99] to generate a charge–current distribution on S_α due solely to bound electric charges. However, this distribution is still too singular to realize. We show how it can be replaced by a simple volume distribution on a spheroidal shell containing S_α . Our method generalizes the usual boundary conditions on an interface between electromagnetic media by allowing the transition to occur gradually, without incurring additional complexity.

We rely on the concepts in [K4], with some improvements in the notation. For further background on physical wavelets and complex-source pulsed beams, see [K94, HLK0, HF1]. The complex distance from the imaginary source point ia to the real observation point r will be denoted by

$$\tilde{r} = \sqrt{(r - ia) \cdot (r - ia)} = p - iq \quad (1)$$

and the complex time by

$$\tilde{t} = t - ib. \quad (2)$$

The imaginary time b plays the role of an overall scale parameter, similar to the scale of ordinary wavelets in one dimension, determining the duration of the pulsed-beam wavelets. The imaginary space vector a similarly controls the spatial extent and orientation of the wavelets. The real and imaginary parts of \tilde{r} satisfy the inequalities

$$|p| \leq r, \quad |q| \leq a \quad \text{where } r = |r|, \quad a = |a|. \quad (3)$$

For fixed $a \neq \mathbf{0}$, the branch points of \tilde{r} in \mathbb{R}^3 form the circle

$$\mathcal{C} = \{r : \tilde{r} = 0\} = \{r : a \cdot r = 0, r = a\}$$

and the ‘standard’ branch of \tilde{r} , defined by $p \geq 0$, has for its branch cut the disc

$$\mathcal{D} = \{r : p = 0\} = \{r : a \cdot r = 0, r \leq a\}.$$

Here \tilde{r} is real-analytic in \mathbb{R}^3 except for a jump discontinuity due to a sign reversal upon crossing \mathcal{D} . Every other branch satisfying the positivity condition

$$a \rightarrow \mathbf{0} \Rightarrow \tilde{r} \rightarrow r \geq 0$$

can be obtained by continuously deforming \mathcal{D} to a membrane \mathcal{B} with the same boundary,

$$\partial\mathcal{B} = \partial\mathcal{D} = \mathcal{C}.$$

The associated branch of \tilde{r} is defined by

$$\tilde{r}_B = p_B - iq_B = \begin{cases} \tilde{r}, & r \notin V_B \\ \tilde{r}, & r \in V_B \end{cases} \quad (4)$$

where V_B is the compact volume swept out by deforming \mathcal{D} to \mathcal{B} . It follows [K4a] that \tilde{r}_B is real-analytic in \mathbb{R}^3 except for a sign reversal across \mathcal{B} .

The scalar wavelet¹ with branch cut \mathcal{B} is defined by

$$\Psi_B = \frac{\tilde{g}(\tilde{t} - \tilde{r}_B)}{\tilde{r}_B},$$

where \tilde{g} is the ‘analytic signal’ associated with a driving signal $g(t)$ exciting the source by

$$\tilde{g}(\tilde{t}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(t') dt'}{\tilde{t} - t'}, \quad (5)$$

which is indeed analytic in the complement of the support of g :

$$\frac{\partial \tilde{g}(\tilde{t})}{\partial \tilde{t}^*} = \frac{1}{2} (\partial_t - i\partial_b) \tilde{g}(t - ib) = 0 \quad \forall \tilde{t} \notin \text{supp } g \subset \mathbb{R}. \quad (6)$$

The significance of the extension parameter b can be understood by noting that the real and imaginary parts of \tilde{g} are

$$g_b(t) = \frac{b}{2\pi} \int_{-\infty}^{\infty} \frac{g(t') dt'}{(t - t')^2 + b^2}, \quad \hat{g}_b(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(t' - t)g(t') dt'}{(t - t')^2 + b^2}. \quad (7)$$

¹ Strictly speaking, the term ‘wavelet’ should be reserved for certain choices of g , as explained in [K4]. The scalar wavelet of order n is obtained with \tilde{g} as the n th derivative of the Cauchy kernel $C(\tilde{t}) = 1/2\pi i\tilde{t}$. Also, note that we use units in which the propagation speed is $c = 1$.

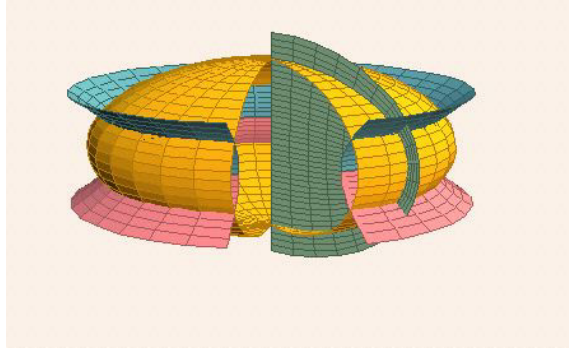


Figure 1. (p, q, ϕ) form an oblate spheroidal coordinate system.

Here g_b is a smoothed version of g with b as the scale or resolution parameter, while \hat{g}_b is a smoothed version of the Hilbert transform of g , again with b as the scale parameter. Thus, time variations of order less than $|b|$ are suppressed in $\tilde{g}(t - ib)$.

Due to its denominator, $\Psi_{\mathcal{B}}$ is singular on \mathcal{C} , where $\tilde{r}_{\mathcal{B}} = 0$, and discontinuous in the interior of \mathcal{B} , where $\tilde{r}_{\mathcal{B}}$ reverses sign. To avoid any further singularities, we want to ensure that the numerator $\tilde{g}(\tilde{t} - \tilde{r}_{\mathcal{B}})$ is analytic in all of \mathbb{R}^3 , and for this it suffices to have its argument bounded away from the real axis by a positive distance. Since

$$\tilde{t} - \tilde{r}_{\mathcal{B}} = t - p_{\mathcal{B}} - i(b - q_{\mathcal{B}}) \quad (8)$$

and $|q_{\mathcal{B}}| = |q| \leq a$, a necessary and sufficient condition is

$$a < |b|. \quad (9)$$

This states that the imaginary spacetime four-vector (a, b) is time-like, belonging to the future cone of spacetime if $a < b$ and the past cone if $b < -a$. The condition (9) will be assumed from now on, making $\Psi_{\mathcal{B}}$ real-analytic in $\mathbb{R}^3 - \mathcal{B}$.

The source $\Sigma_{\mathcal{B}}$ of $\Psi_{\mathcal{B}}$ is defined by applying the wave operator:

$$4\pi \Sigma_{\mathcal{B}} = \square \Psi_{\mathcal{B}} \quad \text{where} \quad \square = \partial_t^2 - \Delta. \quad (10)$$

$\Sigma_{\mathcal{B}}$ can be easily shown to vanish wherever $\Psi_{\mathcal{B}}$ is twice differentiable, hence

$$r \notin \mathcal{B} \Rightarrow \Sigma_{\mathcal{B}} = 0. \quad (11)$$

To characterize the source on \mathcal{B} , we must apply \square in a distributional sense [GS64]. Just as differentiating the Heaviside function gives the delta-function, differentiating a discontinuous function like $\Psi_{\mathcal{B}}$ in a distributional sense gives a single layer on the surface of discontinuity, represented by a delta-function of a variable normal to that surface. Since $\Sigma_{\mathcal{B}}$ is obtained by differentiating $\Psi_{\mathcal{B}}$ twice, it will consist of a combination of single and double layers on \mathcal{B} [K4, K4a]. Moreover, these layers diverge on the boundary $\mathcal{C} = \partial\mathcal{B}$ since $\Psi_{\mathcal{B}}$ is singular there. This singularity will be tamed below by combining wavelets with different branch cuts.

The variables (p, q) defined by (1), together with the azimuthal angle ϕ about the \mathbf{a} -axis, determine an oblate spheroidal coordinate system where the level surfaces of p are the spheroids

$$\mathcal{S}_p : \frac{x^2 + y^2}{p^2 + a^2} + \frac{z^2}{p^2} = 1, \quad p \neq 0 \quad (12)$$

and the level surfaces of q are the orthogonal hyperboloids \mathcal{H}_q . All these quadrics are confocal, having the circle \mathcal{C} as their common focal set. This is depicted in figure 1. As $p \rightarrow 0$, \mathcal{S}_p shrinks to a double cover of the disc \mathcal{D} .

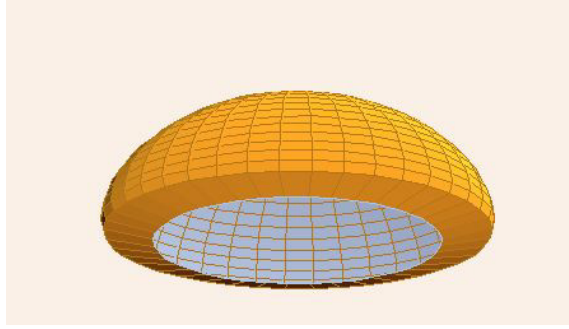


Figure 2. The upper hemispheroidal branch cut \mathcal{B}_α^+ with its apron \mathcal{A}_α .

When the source of Ψ_B is computed, it will be singular on \mathcal{C} due to the singularity of Ψ_B there. In the case of a real point source, this corresponds to the singularity of $\delta(r)$ at $r = \mathbf{0}$. Recall that the latter can be regularized by replacing the origin with a sphere of small radius $r = \alpha$, whence $\delta(r)$ is replaced by a uniform distribution on the spherical surface. The delta-function can then be defined in terms of the limit $\alpha \rightarrow 0$. The equivalent procedure now is to replace the sphere by the oblate spheroid \mathcal{S}_α , which is defined by $p = \text{Re } \tilde{r} = \alpha > 0$. But we can go a step further and represent \mathcal{S}_α as a sum of two branch cuts, something that cannot be done for a real point source since the deformation of a point is still a point. Thus, consider the branch cut

$$\mathcal{B}_\alpha^+ = \mathcal{S}_\alpha^+ \cup \mathcal{A}_\alpha \quad (13)$$

consisting of union of the upper hemispheroid

$$\mathcal{S}_\alpha^+ = \{r \in \mathcal{S}_\alpha : z > 0\}$$

and the apron

$$\mathcal{A}_\alpha = \{r : r \cdot a = 0, a^2 \leq r^2 \leq a^2 + \alpha^2\}$$

connecting \mathcal{S}_α^+ to the branch circle \mathcal{C} , as shown in figure 2. (The apron must be included so that $\partial\mathcal{B}_\alpha^+ = \mathcal{C}$ as required.)

Similarly, let

$$\mathcal{B}_\alpha^- = \mathcal{S}_\alpha^- \cup \mathcal{A}_\alpha \quad (14)$$

be the union of the lower hemispheroid

$$\mathcal{S}_\alpha^- = \{r \in \mathcal{S}_\alpha : z < 0\}$$

with \mathcal{A}_α . For simplicity, denote the complex distance with branch cut \mathcal{B}_α^\pm by \tilde{r}_\pm instead of $\tilde{r}_{\mathcal{B}_\alpha^\pm}$ and the corresponding wavelet by Ψ_\pm . Let V_α^\pm be the interiors of the upper and lower hemispheroids and V_α be the interior of \mathcal{S}_α . According to (4),

$$\begin{aligned} r \in V_\alpha^+ &\Rightarrow \Psi_+(\tilde{r}, \tilde{t}) = \Psi(-\tilde{r}, \tilde{t}), \quad \Psi_-(\tilde{r}, \tilde{t}) = \Psi(\tilde{r}, \tilde{t}) \\ r \in V_\alpha^- &\Rightarrow \Psi_+(\tilde{r}, \tilde{t}) = \Psi(\tilde{r}, \tilde{t}), \quad \Psi_-(\tilde{r}, \tilde{t}) = \Psi(-\tilde{r}, \tilde{t}) \\ r \notin V_\alpha &\Rightarrow \Psi_+(\tilde{r}, \tilde{t}) = \Psi_-(\tilde{r}, \tilde{t}) = \Psi(\tilde{r}, \tilde{t}). \end{aligned}$$

Consider the average of Ψ_\pm ,

$$\Psi_A(\tilde{r}, \tilde{t}) = \frac{1}{2}\{\Psi_-(\tilde{r}, \tilde{t}) + \Psi_+(\tilde{r}, \tilde{t})\}. \quad (15)$$

Then by the above,

$$\Psi_A(\tilde{r}, \tilde{t}) = \begin{cases} \Psi_1(\tilde{r}, \tilde{t}), & r \in V_\alpha \\ \Psi_2(\tilde{r}, \tilde{t}), & r \notin V_\alpha \end{cases} \quad (16)$$

where the internal field Ψ_1 and the external field Ψ_2 are

$$\Psi_1(\tilde{r}, \tilde{t}) = \frac{1}{2}\{\Psi(\tilde{r}, \tilde{t}) + \Psi(-\tilde{r}, \tilde{t})\}, \quad \Psi_2(\tilde{r}, \tilde{t}) = \Psi(\tilde{r}, \tilde{t}) \quad (17)$$

and will, for later purposes, be regarded as functions on all of \mathbb{R}^3 . It follows directly from the definition (5) of \tilde{g} that

$$\begin{aligned} \Psi_1(\tilde{r}, \tilde{t}) &= \frac{1}{4\pi i \tilde{r}} \int_{-\infty}^{\infty} g(t') dt' \left\{ \frac{1}{\tilde{t} - t' - \tilde{r}} - \frac{1}{\tilde{t} - t' + \tilde{r}} \right\} \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(t') dt'}{(\tilde{t} - t')^2 - \tilde{r}^2}, \end{aligned} \quad (18)$$

which depends only on \tilde{r}^2 and is therefore independent of the choice of branch cut. Furthermore, since (9) ensures that

$$|\tilde{t} - t' \pm \tilde{r}| \geq |\operatorname{Im}(\tilde{t} - t' \pm \tilde{r})| = |b \pm q| \geq |b| - a > 0,$$

we have

$$|\tilde{t} - t'|^2 - \tilde{r}^2 > (|b| - a)^2 > 0 \quad \forall r.$$

This shows that Ψ_1 is real-analytic in \mathbb{R}^3 , at least if g has compact support or decays sufficiently rapidly to ensure that the integral (18) converges. That is, by taking the average (17) we have managed to cancel the singularities of $\Psi(\pm\tilde{r}, \tilde{t})$ on \mathcal{C} as well as their jump discontinuities across \mathcal{D} , leaving a field which is analytic in all of \mathbb{R}^3 and hence sourceless:

$$\square\Psi_1(\tilde{r}, \tilde{t}) = 0 \quad \forall(r, t) \in \mathbb{R}^4.$$

The only region where Ψ_A fails to be analytic is therefore \mathcal{S}_α , where it is discontinuous by (16). But even there, the irregularity is mild in the sense that the jump discontinuity

$$\Psi_J(\tilde{r}, \tilde{t}) \equiv \Psi_2(\tilde{r}, \tilde{t}) - \Psi_1(\tilde{r}, \tilde{t}) = \frac{1}{2}\{\Psi(\tilde{r}, \tilde{t}) - \Psi(-\tilde{r}, \tilde{t})\} \quad (19)$$

is bounded—unlike that in any single branch Ψ_B , which is singular on \mathcal{C} . Like Ψ_1 and Ψ_2 , Ψ_J will be regarded as a field on all of \mathbb{R}^3 , although for the present we need it only on \mathcal{S}_α . The source Σ_A of Ψ_A , defined by

$$4\pi \Sigma_A = \square\Psi_A, \quad (20)$$

is therefore a distribution supported on \mathcal{S}_α . By the argument below (10), it consists of a combination of single and double layers on \mathcal{S}_α , with the difference that these layers are now bounded since \mathcal{S}_α avoids the singular circle \mathcal{C} . Even so, it is not clear that the double layer can be realized in practice. In the next section we replace Ψ_A by a continuous field, where the transition from Ψ_1 to Ψ_2 occurs gradually over a range of spheroids, whose source is supported on a spheroidal shell instead of a single spheroid. (This explains the reason for viewing Ψ_1 , Ψ_2 and Ψ_J as fields on all of \mathbb{R}^3 .) Such volume sources, and their electromagnetic counterparts considered in the following section, should be realizable.

2. Spheroidal shell sources

Let H be the Heaviside step function. Since $0 \leq p < \alpha$ in the interior of S_α and $p > \alpha$ in the exterior, we have

$$\Psi_A = H(\alpha - p)\Psi_1 + H(p - \alpha)\Psi_2. \quad (21)$$

It is natural to use the vector field orthogonal to S_p given by [K3, appendix]

$$\mathbf{n} \equiv \nabla p = \frac{pr + qa}{p^2 + q^2}, \quad (22)$$

which is unnormalized with

$$n^2 \equiv |\mathbf{n}|^2 = \frac{p^2 + a^2}{p^2 + q^2} \geq 1, \quad \mathbf{n} \cdot \nabla q = 0, \quad \nabla \cdot \mathbf{n} = \frac{2p}{p^2 + q^2}. \quad (23)$$

We can compute the source Σ_A in (20) by using

$$\nabla H(p - \alpha) = -\nabla H(\alpha - p) = H'(p - \alpha)\mathbf{n} = \delta(p - \alpha)\mathbf{n}. \quad (24)$$

Applying the wave operator gives Σ_A as a combination of terms involving $\delta(p - \alpha)$, interpreted as single layers on S_α , and $\delta'(p - \alpha)$, interpreted as *double layers*. As mentioned above, it is doubtful whether the double layer can be realized in practice, and this will get still worse in the electromagnetic case, where the currents involve one more derivative. Since we are interested in physically realizable sources, we now proceed to modify the above construction. The terms involving $\delta(p - \alpha)$ and $\delta'(p - \alpha)$ are unavoidable as long as the source is confined to the surface S_α . To construct more realistic sources, we now choose a function h that approximates the Heaviside function. A convenient example is

$$h(p) = \frac{1}{\pi} \arg(-p + i\varepsilon) = \frac{1}{\pi} \operatorname{Im} \operatorname{Log}(-p + i\varepsilon), \quad \varepsilon > 0 \quad (25)$$

which becomes the Heaviside function in the limit as $\varepsilon \rightarrow 0$. Note that

$$h'(p) = \frac{\varepsilon}{\pi(p^2 + \varepsilon^2)} \quad (26)$$

is indeed an approximation to the delta-function. But the simple choice (25) has the drawback that h' is not compactly supported, so the resulting source, although extremely small outside a shell of thickness $\mathcal{O}(\varepsilon)$, will not have strictly compact spatial support. To obtain a compactly supported source, we now assume that h has the following properties,

$$h(p) + h(-p) = 1 \quad \text{and} \quad h(p) = \begin{cases} 0, & p \leq -\varepsilon \\ 1, & p \geq \varepsilon \end{cases} \quad (27)$$

where ε will be assumed fixed with

$$0 < \varepsilon < \alpha, \quad (28)$$

so that $h(p - \alpha)$ vanishes on the disc \mathcal{D} where $p = 0$. Define the regularized version of Ψ_A by replacing H by h in (21),

$$\Psi_A^\varepsilon = h(\alpha - p)\Psi_1(\tilde{r}, \tilde{t}) + h(p - \alpha)\Psi_2(\tilde{r}, \tilde{t}). \quad (29)$$

To simplify the equations, we use the abbreviations

$$h_1(p) = h(\alpha - p), \quad h_2(p) = h(p - \alpha), \quad (30)$$

so that

$$\Psi_A^\varepsilon = h_1\Psi_1 + h_2\Psi_2 \equiv h_k\Psi_k \quad (31)$$

where the Einstein convention of summing over repeated indices has been used. The source Σ_A^ε of Ψ_A^ε is defined as usual by

$$4\pi \Sigma_A^\varepsilon \equiv \square \Psi_A^\varepsilon. \quad (32)$$

To compute this, note that since (27) implies $h'(-p) = h'(p)$,

$$\begin{aligned} \nabla \Psi_A^\varepsilon &= -h'(\alpha - p)\Psi_1 \mathbf{n} + h'(p - \alpha)\Psi_2 \mathbf{n} + h_k \nabla \Psi_k \\ &= h'(p - \alpha)\Psi_J \mathbf{n} + h_k \nabla \Psi_k \end{aligned} \quad (33)$$

where Ψ_J , defined as in (19) and given by

$$\begin{aligned} \Psi_J(\tilde{r}, \tilde{t}) &= \frac{1}{4\pi i\tilde{r}} \int_{-\infty}^{\infty} g(t') dt' \left\{ \frac{1}{\tilde{t} - t' - \tilde{r}} + \frac{1}{\tilde{t} - t' + \tilde{r}} \right\} \\ &= \frac{1}{2\pi i\tilde{r}} \int_{-\infty}^{\infty} \frac{(\tilde{t} - t')g(t') dt'}{(\tilde{t} - t')^2 - \tilde{r}^2}, \end{aligned} \quad (34)$$

no longer represents a jump discontinuity of the field since we are not confined to a single spheroid. By the same argument used to show that Ψ_1 is real-analytic in \mathbb{R}^3 , it follows that Ψ_J is real-analytic in \mathbb{R}^3 except for being discontinuous on \mathcal{D} and singular on \mathcal{C} due to the factor $1/\tilde{r}$. Taking the divergence of (33) gives

$$\Delta \Psi_A^\varepsilon = h''(p - \alpha)\Psi_J n^2 + 2h'(p - \alpha)\nabla \Psi_J \cdot \mathbf{n} + h'(p - \alpha)\Psi_J \nabla \cdot \mathbf{n} + h_k \nabla \Psi_k.$$

Since $\Psi_J(\tilde{r}, \tilde{t})$ is complex-analytic in \tilde{r} when $r \notin \mathcal{D}$,

$$p > 0 \Rightarrow \nabla \Psi_J = \Psi_{J'} \nabla \tilde{r}$$

where the prime denotes the complex derivative with respect to \tilde{r} ,

$$\Psi_{J'} = \frac{\partial \Psi_J}{\partial \tilde{r}} = \frac{1}{2}(\partial_p + i\partial_q)\Psi_J = \frac{1}{2}\{\Psi'(\tilde{r}, \tilde{t}) + \Psi'(-\tilde{r}, \tilde{t})\}. \quad (35)$$

By (23),

$$\nabla \tilde{r} \cdot \mathbf{n} = (\nabla p - i\nabla q) \cdot \mathbf{n} = n^2.$$

Subtracting $\partial_t^2 \Psi_A^\varepsilon$ thus gives

$$-4\pi \Sigma_A^\varepsilon = h''(p - \alpha)\Psi_J n^2 + 2h'(p - \alpha)\Psi_{J'} n^2 + h'(p - \alpha)\Psi_J \nabla \cdot \mathbf{n} - h_k \square \Psi_k.$$

But we have seen that $\square \Psi_1$ vanishes identically and $\square \Psi_2$ is supported on \mathcal{D} , where $h_2 = 0$ by (27) and (28). Using (23) gives the regularized source

$$\boxed{-4\pi \Sigma_A^\varepsilon = h''(p - \alpha)\Psi_J n^2 + \frac{2h'(p - \alpha)}{p^2 + q^2}\{(p^2 + a^2)\Psi_{J'} + p\Psi_J\}} \quad (36)$$

supported on the spheroidal shell

$$\mathcal{S}_\alpha^\varepsilon = \{r : \alpha - \varepsilon \leq p \leq \alpha + \varepsilon\}. \quad (37)$$

We emphasize that Σ_A^ε is a smooth volume source that depends only on the ‘jump field’ Ψ_J . Taking the limit $\varepsilon \rightarrow 0$ so that h becomes the Heaviside function gives the source Σ_A consisting of single and double layers on \mathcal{S}_α .

3. Maxwell's equations and Hertz potentials

We work with the following complex combinations of electromagnetic fields:

$$\mathbf{F} = \mathbf{D} + i\mathbf{B} \quad (38)$$

$$\mathbf{G} = \mathbf{E} + i\mathbf{H} = \mathbf{F} - 4\pi\mathbf{P} \quad (39)$$

$$\mathbf{P} = \mathbf{P}_e + i\mathbf{P}_m \quad (40)$$

where the units are Gaussian with $c = 1$, \mathbf{P}_e is the electric dipole density, and \mathbf{P}_m is the magnetic dipole density. Maxwell's equations take the form

$$\nabla \cdot \mathbf{F} = 4\pi\rho \quad \dot{\mathbf{F}} + i\nabla \times \mathbf{G} = -4\pi\mathbf{J} \quad (41)$$

where $\dot{\mathbf{F}} = \partial_t \mathbf{F}$. In the general case of complex charge and current densities

$$\rho = \rho_e + i\rho_m \quad \mathbf{J} = \mathbf{J}_e + i\mathbf{J}_m,$$

equations (41) are equivalent to

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 4\pi\rho_e & \dot{\mathbf{D}} - \nabla \times \mathbf{H} &= -4\pi\mathbf{J}_e \\ \nabla \cdot \mathbf{B} &= 4\pi\rho_m & \dot{\mathbf{B}} + \nabla \times \mathbf{E} &= -4\pi\mathbf{J}_m, \end{aligned} \quad (42)$$

so the imaginary parts (ρ_m, \mathbf{J}_m) represent the magnetic charge–current density. Since magnetic monopoles are not observed, we must require

$$\rho_m = 0 \quad \text{and} \quad \mathbf{J}_m = \mathbf{0}. \quad (43)$$

That is, equation (41) are completely equivalent to the usual Maxwell equations if we add the requirement that (ρ, \mathbf{J}) is real. We will consider solutions derived from a complex *Hertz potential* consisting of electric and magnetic Hertz vectors [BW99, J99]

$$\mathbf{Z} = \mathbf{Z}_e + i\mathbf{Z}_m \quad (44)$$

whose source is the polarization,

$$\square \mathbf{Z} = 4\pi\mathbf{P}. \quad (45)$$

(For this reason, $(\mathbf{Z}_e, \mathbf{Z}_m)$ are sometimes called polarization potentials.) The field \mathbf{F} is then given in terms of \mathbf{Z} by

$$\mathbf{F} = \nabla \times \nabla \times \mathbf{Z} + i\nabla \times \dot{\mathbf{Z}}, \quad (46)$$

and it follows from (39) and (45) that

$$\begin{aligned} \mathbf{G} &= \mathbf{F} - \square \mathbf{Z} = \nabla \times \nabla \times \mathbf{Z} + i\nabla \times \dot{\mathbf{Z}} + \Delta \mathbf{Z} - \ddot{\mathbf{Z}} \\ &= \nabla \nabla \cdot \mathbf{Z} + i\nabla \times \dot{\mathbf{Z}} - \ddot{\mathbf{Z}}. \end{aligned} \quad (47)$$

The real form of equations (45)–(47) is [BW99, pp 84, 85]

$$\begin{aligned} \square \mathbf{Z}_e &= 4\pi\mathbf{P}_e & \square \mathbf{Z}_m &= 4\pi\mathbf{P}_m \\ \mathbf{E} &= \nabla \nabla \cdot \mathbf{Z}_e - \nabla \times \dot{\mathbf{Z}}_m - \ddot{\mathbf{Z}}_e & \mathbf{B} &= \nabla \times \nabla \times \mathbf{Z}_m + \nabla \times \dot{\mathbf{Z}}_e \\ \mathbf{H} &= \nabla \nabla \cdot \mathbf{Z}_e + \nabla \times \dot{\mathbf{Z}}_e - \ddot{\mathbf{Z}}_m & \mathbf{D} &= \nabla \times \nabla \times \mathbf{Z}_e - \nabla \times \dot{\mathbf{Z}}_m. \end{aligned}$$

An inspection of the expressions for \mathbf{E} and \mathbf{B} reveals the meaning of the Hertz potentials as ‘superpotentials’ from which the four-vector potential (Φ, \mathbf{A}) can be derived by

$$\Phi = -\nabla \cdot \mathbf{Z}_e \quad \mathbf{A} = \nabla \times \mathbf{Z}_m + \dot{\mathbf{Z}}_e. \quad (48)$$

In fact, these automatically satisfy the Lorenz condition

$$\dot{\Phi} + \nabla \cdot \mathbf{A} = 0,$$

and every four-vector potential satisfying it can be derived from Hertz potentials. The freedom to choose a gauge for (Φ, \mathbf{A}) , including a non-Lorenz gauge, is part of a much greater gauge freedom in $(\mathbf{Z}_e, \mathbf{Z}_m)$ [N55, BW99].

According to (46), \mathbf{F} is a curl, so by (41) the free charge density vanishes:

$$\rho = 0. \quad (49)$$

Furthermore, (47) gives

$$\nabla \times \mathbf{G} = i\nabla \times \nabla \times \dot{\mathbf{Z}} - \nabla \times \ddot{\mathbf{Z}} = i\dot{\mathbf{F}},$$

therefore by (41), the free current density also vanishes:

$$\mathbf{J} = \mathbf{0}. \quad (50)$$

Maxwell's equations (42), written in terms of the microscopic fields (\mathbf{E}, \mathbf{B}) , now state that

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{E} + \dot{\mathbf{B}} &= \mathbf{0} \\ \nabla \cdot \mathbf{E} &= 4\pi\rho_b & \nabla \times \mathbf{B} - \dot{\mathbf{E}} &= 4\pi\mathbf{J}_b \end{aligned} \quad (51)$$

where

$$\rho_b = -\nabla \cdot \mathbf{P}_e \quad \mathbf{J}_b = \dot{\mathbf{P}}_e + \nabla \times \mathbf{P}_m \quad (52)$$

represent the bound charge and current densities generated by the variable polarizations $(\mathbf{P}_e, \mathbf{P}_m)$. The fields derived from Hertz potentials as above are thus due entirely to bound sources².

4. Spheroidal electromagnetic antennas

In this section, we construct electromagnetic wavelets from scalar wavelets by turning Ψ into \mathbf{Z} , then compute their charge–current densities. It is essential that the polarization \mathbf{P} defined in (45) have compact spatial support, as it cannot otherwise be realized. There are various ways to turn a scalar solution of the wave equation into a vector solution without increasing the support of its source distribution, the simplest being

$$\mathbf{Z} = p\Psi \quad (53)$$

where p is a constant (possibly complex) vector. The polarization is then given by

$$4\pi\mathbf{P} = p\nabla^2\Psi = 4\pi p\Sigma, \quad (54)$$

so \mathbf{P} and Σ have the same support. Since Σ is a distribution consisting of single and double layers on \mathcal{D} , so is \mathbf{P} . A similar construction applies to the different versions supported on the general branch cut \mathcal{B} and the spheroid \mathcal{S}_α . As explained below (24), the layers on \mathcal{B} are singular on \mathcal{C} while those on \mathcal{S}_α are bounded. Even so, the charge–current distributions (52) require one further differentiation, hence they generate a still higher layer with coefficient $\delta''(p - \alpha)$, and it is doubtful whether such distributions can be realized. For this reason, we confine our analysis to volume sources on the spheroidal shell $\mathcal{S}_\alpha^\varepsilon$ (37). Define the Hertz potential

$$\mathbf{Z}_A^\varepsilon = p\Psi_A^\varepsilon \quad (55)$$

with Ψ_A^ε as in (29), whose polarization density is

$$\mathbf{P}_A^\varepsilon = p\Sigma_A^\varepsilon \quad (56)$$

² Free charge–current densities can be added by using stream potentials [N55].

with Σ_A^ε given by (36). If we interpret Σ_A^ε as a scalar density, then (56) suggests an interpretation of \mathbf{p} as a (complex) combination of electric and magnetic dipole moments. The charge and current densities on the shell, as given by (52), are

$$\rho_b = -\operatorname{Re}\{\nabla \cdot \mathbf{P}_A^\varepsilon\} = -\operatorname{Re}\{\mathbf{p} \cdot \nabla \Sigma_A^\varepsilon\} \quad (57)$$

and

$$\mathbf{J}_b = \operatorname{Re}\{\dot{\mathbf{P}}_A^\varepsilon\} + \operatorname{Im}\{\nabla \times \mathbf{P}_A^\varepsilon\} = \operatorname{Re}\{\mathbf{p} \dot{\Sigma}_A^\varepsilon\} - \operatorname{Im}\{\mathbf{p} \times \nabla \Sigma_A^\varepsilon\}. \quad (58)$$

Outside the shell $\mathcal{S}_\alpha^\varepsilon$ the potential Z_A^ε coincides with $Z = \mathbf{p}\Psi$, whose pulsed-beam field \mathbf{F} was computed in [K4].

5. Extended Huygens sources

The above suggests an generalization of Huygens sources [Hy99], allowing equivalent sources to be represented on shells instead of surfaces surrounding a bounded source. We present this generalization and compare it to the usual method based on boundary conditions on an interface between electromagnetic media. Let $p(\mathbf{r}, t)$ be a differentiable function, which will be called a zone function. Fix two numbers $p_1 < p_2$ and consider the time-dependent surfaces and volumes in \mathbb{R}^3 defined by

$$\begin{aligned} S_1(t) &= \{\mathbf{r} : p(\mathbf{r}, t) = p_1\}, & S_2(t) &= \{\mathbf{r} : p(\mathbf{r}, t) = p_2\} \\ V_1(t) &= \{\mathbf{r} : p(\mathbf{r}, t) < p_1\}, & V_2(t) &= \{\mathbf{r} : p(\mathbf{r}, t) > p_2\}. \end{aligned}$$

Given two electromagnetic fields $(\mathbf{F}_1, \mathbf{G}_1)$ and $(\mathbf{F}_2, \mathbf{G}_2)$, with or without sources, we want to construct an interpolated field (\mathbf{F}, \mathbf{G}) so that

$$\begin{aligned} \mathbf{F}(\mathbf{r}, t) &= \begin{cases} \mathbf{F}_1(\mathbf{r}, t), & \mathbf{r} \in V_1(t) \\ \mathbf{F}_2(\mathbf{r}, t), & \mathbf{r} \in V_2(t) \end{cases} \\ \mathbf{G}(\mathbf{r}, t) &= \begin{cases} \mathbf{G}_1(\mathbf{r}, t), & \mathbf{r} \in V_1(t) \\ \mathbf{G}_2(\mathbf{r}, t), & \mathbf{r} \in V_2(t). \end{cases} \end{aligned} \quad (59)$$

Choose a differentiable function $h_2(\mathbf{r}, t)$ such that

$$h_2(\mathbf{r}, t) = \begin{cases} 0, & \mathbf{r} \in V_1(t) \\ 1, & \mathbf{r} \in V_2(t) \end{cases} \quad (60)$$

and let

$$h_1(\mathbf{r}, t) = 1 - h_2(\mathbf{r}, t).$$

We define the interpolated field as

$$\begin{aligned} \mathbf{F}(\mathbf{r}, t) &= h_k(\mathbf{r}, t) \mathbf{F}_k(\mathbf{r}, t) \\ \mathbf{G}(\mathbf{r}, t) &= h_k(\mathbf{r}, t) \mathbf{G}_k(\mathbf{r}, t) \end{aligned} \quad (61)$$

where summations over $k = 1, 2$ are implied, and the jump field

$$\begin{aligned} \mathbf{F}_J &= \mathbf{F}_2 - \mathbf{F}_1 = \mathbf{D}_J + i\mathbf{B}_J \\ \mathbf{G}_J &= \mathbf{G}_2 - \mathbf{G}_1 = \mathbf{E}_J + i\mathbf{H}_J. \end{aligned} \quad (62)$$

Then, according to (41), the charge density of (\mathbf{F}, \mathbf{G}) is

$$4\pi\rho = \nabla \cdot \mathbf{F} = 4\pi h_k \rho_k + \nabla h_2 \cdot \mathbf{F}_J \quad (63)$$

where

$$4\pi\rho_k = \nabla \cdot \mathbf{F}_k, \quad k = 1, 2 \quad (64)$$

are the charge densities of the prescribed fields. Thus, in addition to the interpolated charge density

$$\rho_I = h_k \rho_k \quad (65)$$

we have a transitional charge density given by

$$4\pi \rho_T = \nabla h_2 \cdot \mathbf{F}_J \quad (66)$$

which depends only on the component of the jump field \mathbf{F}_J parallel to ∇h_2 . According to (60), ρ_T vanishes outside the transition shell

$$V_T(t) = \{\mathbf{r} : p_1 \leq p(\mathbf{r}, t) \leq p_2\}. \quad (67)$$

Similarly, the current density is

$$4\pi \mathbf{J} = -\dot{\mathbf{F}} - i\nabla \times \mathbf{G} = 4\pi h_k \mathbf{J}_k - \dot{h}_2 \mathbf{F}_J - i\nabla h_2 \times \mathbf{G}_J \quad (68)$$

where

$$4\pi \mathbf{J}_k = -\dot{\mathbf{F}}_k - i\nabla \times \mathbf{G}_k, \quad k = 1, 2$$

are the current densities of the prescribed fields. Hence \mathbf{J} is the sum of the interpolated current density

$$\mathbf{J}_I = h_k \mathbf{J}_k \quad (69)$$

and a transitional current density on $V_T(t)$ given by

$$4\pi \mathbf{J}_T = -\dot{h}_2 \mathbf{F}_J - i\nabla h_2 \times \mathbf{G}_J \quad (70)$$

which depends only on \mathbf{F}_J (if h_2 is time dependent) and the component of \mathbf{G}_J orthogonal to ∇h_2 . The electric and magnetic transitional sources are obtained by taking real and imaginary parts. Assuming h_2 is real, this gives

$$\begin{aligned} 4\pi \rho_T^e &= \nabla h_2 \cdot \mathbf{D}_J & 4\pi \mathbf{J}_T^e &= -\dot{h}_2 \mathbf{D}_J + \nabla h_2 \times \mathbf{H}_J \\ 4\pi \rho_T^m &= \nabla h_2 \cdot \mathbf{B}_J & 4\pi \mathbf{J}_T^m &= -\dot{h}_2 \mathbf{B}_J - \nabla h_2 \times \mathbf{E}_J. \end{aligned} \quad (71)$$

Letting h_2 be complex in $V_T(t)$ makes the transition shell a chiral medium mixing electric and magnetic fields. A further generalization is obtained by replacing h_k with 3×3 matrices (dyadics) \mathbb{H}_k satisfying

$$\mathbb{H}_2(\mathbf{r}, t) = \begin{cases} 0, & \mathbf{r} \in V_1(t) \\ \mathbb{I}, & \mathbf{r} \in V_2(t) \end{cases}, \quad \mathbb{H}_1(\mathbf{r}, t) = \mathbb{I} - \mathbb{H}_2(\mathbf{r}, t) \quad (72)$$

where \mathbb{I} is the unit matrix. This makes the transition shell $V_T(t)$ a *non-isotropic* medium as well as chiral if \mathbb{H}_k are complex. See [LSTV94] for a treatment of chiral and non-isotropic media.

Choosing the zone function $p(\mathbf{r}, t)$ time dependent thus gives a simple formulation of the transition shell as a moving source, which could be useful in the analysis of radiation by moving objects.

To see how all this relates to Huygens' principle, suppose that we are only given a field $(\mathbf{F}_2, \mathbf{G}_2)$ whose charge-current density (ρ_2, \mathbf{J}_2) is confined to $V_1(t)$, and we want to find an equivalent charge-current density confined to $V_T(t)$ whose radiated field in $V_2(t)$ (but not necessarily elsewhere) is $(\mathbf{F}_2, \mathbf{G}_2)$. We are free to choose the field $(\mathbf{F}_1, \mathbf{G}_1)$ in any way that gives vanishing interpolated sources

$$\rho_I = h_k \rho_k = 0, \quad \mathbf{J}_I = h_k \mathbf{J}_k = \mathbf{0}, \quad (73)$$

since the sources of the interpolated field are then purely transitional and hence confined to $V_T(t)$ as desired. To satisfy (73), it suffices to require that (ρ_1, \mathbf{J}_1) be confined to $V_2(t)$.

Thus, choosing any field $(\mathbf{F}_1, \mathbf{G}_1)$ with sources in $V_2(t)$ and any function h_2 satisfying (60), an equivalent charge–current density on $V_T(t)$ is given by (66) and (70). As $p_1 \rightarrow p_2$, $V_T(t)$ becomes S_2 and (ρ_T, \mathbf{J}_T) become ordinary Huygens surface sources [Hy99].

The freedom to choose $(\mathbf{F}_1, \mathbf{G}_1)$ (interpreted as the ‘interior field’ if $V_1(t)$ is bounded) and h_2 is constrained by the requirement that the magnetic charge–current density must vanish, as detailed below.

Now suppose that $p = p(\mathbf{r})$ is time independent, so S_k and V_k are fixed, and choose h_k to be time independent and real. As $p_1 \rightarrow p_2$, assume that

$$\lim_{p_1 \rightarrow p_2} \nabla h_2(\mathbf{r}) = \delta(p(\mathbf{r}) - p_2) \mathbf{n}(\mathbf{r})$$

where $\mathbf{n}(\mathbf{r})$ is a normal vector field on S_2 pointing into V_2 . Then (66) and (70) give

$$\begin{aligned} \rho_T &\rightarrow \delta(p(\mathbf{r}) - p_2) \sigma \\ \mathbf{J}_T &\rightarrow \delta(p(\mathbf{r}) - p_2) \mathbf{K}, \end{aligned} \quad (74)$$

where

$$4\pi \sigma = \mathbf{n} \cdot \mathbf{F}_J \quad \text{and} \quad 4\pi \mathbf{K} = -i \mathbf{n} \times \mathbf{G}_J \quad (75)$$

are the surface charge and current densities on S_2 , whose real and imaginary parts give the electric and magnetic surface sources:

$$\begin{aligned} 4\pi \sigma_e &= \mathbf{n} \cdot \mathbf{D}_J & 4\pi \mathbf{K}_e &= \mathbf{n} \times \mathbf{H}_J \\ 4\pi \sigma_m &= \mathbf{n} \cdot \mathbf{B}_J & 4\pi \mathbf{K}_m &= -\mathbf{n} \times \mathbf{E}_J. \end{aligned} \quad (76)$$

Since magnetic monopoles are not observed, σ_m and \mathbf{K}_m must vanish. When $p_1 < p_2$, this may be accomplished if h_2 can be chosen so that

$$\nabla h_2 \cdot \mathbf{B}_J = 0, \quad \nabla h_2 \times \mathbf{E}_J = \mathbf{0}, \quad (77)$$

which is possible³ if

$$\mathbf{E}_J \cdot \mathbf{B}_J = 0 \quad \forall \mathbf{r} \in V_T(t). \quad (78)$$

In the limit $p_1 \rightarrow p_2$, (77) reduces (76) to

$$\begin{aligned} \mathbf{n} \cdot \mathbf{D}_J &= 4\pi \sigma_e & \mathbf{n} \times \mathbf{H}_J &= 4\pi \mathbf{K}_e \\ \mathbf{n} \cdot \mathbf{B}_J &= 0 & \mathbf{n} \times \mathbf{E}_J &= \mathbf{0}, \end{aligned} \quad (79)$$

which are the usual boundary conditions on an interface between two media.

Returning to the general time-dependent setting, consider now an alternative procedure of special interest here. Instead of interpolating two prescribed fields $(\mathbf{F}_k, \mathbf{G}_k)$, let us interpolate two Hertz potentials \mathbf{Z}_k :

$$\mathbf{Z} = h_k \mathbf{Z}_k. \quad (80)$$

As seen, this automatically results in vanishing ‘free’ sources $\rho = 0$, $\mathbf{J} = \mathbf{0}$. The polarization is found to be

$$4\pi \mathbf{P} = \square \mathbf{Z} = 4\pi h_k \mathbf{P}_k + 2\dot{h}_2 \dot{\mathbf{Z}}_J - 2(\nabla h_2 \cdot \nabla) \mathbf{Z}_J - \mathbf{Z}_J \Delta h_2$$

where $4\pi \mathbf{P}_k = \square \mathbf{Z}_k$ are the polarizations of the prescribed fields. If

$$\text{supp} \mathbf{P}_1 \subset V_2 \quad \text{and} \quad \text{supp} \mathbf{P}_2 \subset V_1,$$

then the interpolated polarization $h_k \mathbf{P}_k$ vanishes and the polarization is purely transitional on $V_T(t)$:

$$4\pi \mathbf{P} = 2\dot{h}_2 \dot{\mathbf{Z}}_J - 2(\nabla h_2 \cdot \nabla) \mathbf{Z}_J - \mathbf{Z}_J \Delta h_2. \quad (81)$$

This generalizes (56), and the bound charge–current densities derived from \mathbf{P}_e and \mathbf{P}_m via (52) generalize (57) and (58).

³ Letting V_T be a non-isotropic medium by using \mathbb{H}_k (72) makes it easier to enforce the absence of magnetic monopoles.

6. Conclusions

We have improved on the computation of sources for electromagnetic wavelets given in [K4] in two ways: (a) the spheroidal surface S_α supporting the sources has been replaced by a spheroidal shell S_α^ε supporting smooth volume sources. This eliminates the multiple layers on S_α which make the sources difficult if not impossible to realize; (b) by deriving the sources from Hertz potentials, we have eliminated the magnetic charge–current density, further facilitating their realizability.

The problem with the magnetic sources in [K4] can be better understood from the current perspective. Let the zone function be $p = \text{Re } \tilde{r}$ and S_k, V_k be as above with $0 < p_1 < p_2$. Let the fields (F_k, G_k) be derived from the Hertz potentials

$$Z_k = p\Psi_k, \quad k = 1, 2$$

with Ψ_k given by (17). Recall that Ψ_k are analytic in (\tilde{r}, \tilde{t}) for $p > 0$. Therefore the fields (F_k, G_k) are analytic in (\tilde{r}, \tilde{t}) for $r \notin \mathcal{D}$, and

$$P_k = \mathbf{0}, \quad G_k = F_k = E_k + iB_k \quad \forall r \notin \mathcal{D}.$$

The jump fields

$$F_J = G_J = E_J + iB_J$$

are also analytic, as is their polarization scalar [K3a]

$$F_J^2 \equiv F_J \cdot F_J = E_J^2 - B_J^2 + 2iE_J \cdot B_J. \quad (82)$$

The condition (78) thus requires the imaginary part of an analytic function to vanish for $r \in V_T$, which implies that F_J^2 vanishes identically. Although every electromagnetic field must have $F_J^2 \rightarrow 0$ in the far zone [B15], fields satisfying $F_J^2 = 0$ globally, called null fields, are rather degenerate. In particular, the electromagnetic wavelet fields are not null and hence cannot fulfil (78). Instead, we have begun with an interpolated Hertz potential (80) and derived bound sources in the transition shell, thus preserving analyticity without invoking the existence of magnetic monopoles.

There is still an unsatisfactory aspect to the polarization (56) and charge–current density (57), (58). Namely, they depend on the fixed vector p and thus do not conform to the spheroidal geometry. This suggests using methods of constructing Z from Ψ other than (53). While (53) is the complex version of Whittaker’s potentials [W4], there are alternatives which do not require a fixed polarization vector, such as Debye potentials; see [BD98] for example. Such alternatives will be considered in future work.

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Note added. After this paper was finished, I learned from Dr Arthur Yaghjian that a similar formulation of the gradual transition between two electromagnetic media has been developed by Lindell *et al* [LTN0]. The present form is somewhat more general in the following respects. (a) The regions and surfaces are allowed to be time dependent. (b) The derivation of generalized Huygens sources in [LTN0] makes the assumption that one of the prescribed fields vanishes, which we find to be unnecessary. This allows extended Huygens sources on V_T with nonvanishing ‘internal’ fields.

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